Non-uniqueness of the $\lambda\Phi^4$-vacuum

Ralf Schützhold[1,*], Ralf Kuhn[1,2], Michael Meyer-Hermann[1], Günter Plunien[1], and Gerhard Soff[1]
[1] *Institut für Theoretische Physik, Technische Universität Dresden, D-01062 Dresden, Germany*
[2] *Max-Planck-Institut für Physik komplexer Systeme, 01187 Dresden, Germany*
[*] *Electronic address :* `schuetz@theory.phy.tu-dresden.de`

(February 1, 2008)

We prove that the massless neutral $\lambda\Phi^4$-theory does not possess a unique vacuum. Based on the Wightman axioms the non-existence of a state which preserves Poincaré and scale invariance is demonstrated non-perturbatively for a non-vanishing self-interaction. We conclude that it is necessary to break the scale invariance in order to define a vacuum state. The renormalized vacuum expectation value of the energy-momentum tensor is derived from the two-point Wightman function employing the point-splitting technique and its relation to the phionic and the scalar condensate is addressed. Possible implications to other self-interacting field theories and to different approaches in quantum field theory are pointed out.

PACS: 11.10.Cd, 11.15.Tk, 11.30.Qc

I. INTRODUCTION

Various quantum field theories supposed to describe fundamental interactions in physics are scale invariant, for instance gauge field theories describing the electromagnetic and the strong interaction. In many cases, however, one may observe a scale dependence of observables, i.e. vacuum expectation values of suitable operators. In order to elucidate the origin of this scale it might be important to examine the corresponding vacuum states. The properties of the vacuum state of self-interacting theories could provide a deeper understanding of the origin of inherent scales. To start investigations in this direction it seems legitimate to consider at first a simple but generic model scenario. As an instructive example we focus on the massless, neutral $\lambda\Phi^4$-theory.

The $\lambda\Phi^4$-theory was studied extensively in the framework of perturbation theory. This approach is based on the vacuum state of the free theory, which is scale invariant in the massless case and independent of the interaction. A modification of the vacuum state due to symmetry breaking induced by the self-interaction is not attainable in perturbation theory. The ground state of the non-interacting Hamiltonian is unique and coincides with the free vacuum state. However, the interacting Hamiltonian does not necessarily possess a unique ground state and thus an analogous identification with the exact vacuum state of the interacting theory does not hold in general. In order to specify this vacuum state it is essential to employ an appropriate non-perturbative treatment of the interaction.

During the last decades non-perturbative techniques have become increasingly important owing to their relevance to QCD. Special attention was devoted to the non-trivial structure of the vacuum. Especially, there are indications for a vacuum degeneracy in the non-Abelian SU(2)-gauge theory [1]. As another example one may investigate the non-linear Liouville model [2], which does not possess a translationally invariant vacuum.

In this paper we would like to advocate ideas along this line of reasonings. It is our main intention to prove a clear assertion concerning the vacuum state in the special case of the $\lambda\Phi^4$-theory. For this purpose we employ the axiomatic approach of Wightman, cf. [3–6].

This article is organized as follows: The Wightman axioms summarized in the appendix are utilized to deduce a proof of the non-uniqueness of the vacuum state of the scale invariant $\lambda\Phi^4$-theory in Section II. In Section III we introduce the non-perturbative vacuum via breaking scale invariance and evaluate the corresponding expectation values. Finally we address some implications of our results.

II. PROOF OF NON-UNIQUENESS

In this Section we provide a general proof for the non-existence of a unique vacuum in the case of the massless and neutral $\lambda\Phi^4$-theory. For this purpose we construct the rather general form of the corresponding two-point Wightman function. Utilizing the non-linear equation of motion we derive expectation values of higher powers of the fields. If we assume that a unique vacuum exists, these expectation values have to vanish. In view of the postulated cyclicity of the vacuum (see the appendix) this zero results in a contradiction for the non-vanishing self-interaction. Consequently, a unique vacuum cannot exist.

A. Classical $\lambda\Phi^4$-theory

The action of a massless neutral scalar field possessing a $\lambda\Phi^4$ self-coupling is given by

$$\mathcal{A} = \int d^4x \, \left(\frac{1}{2}\,(\partial_\mu\Phi)(\partial^\mu\Phi) - \frac{\lambda}{4!}\,\Phi^4\right) \,. \tag{1}$$

This theory exhibits two important symmetries. As every realistic field theory it obeys Poincaré invariance

$$x^\mu \ \to \ L^\mu_{\ \nu} \, x^\nu + a^\mu \, . \tag{2}$$

On the other hand the action remains unchanged under transformations of the following form

$$\begin{aligned} x^\mu &\to \Omega^{-1} x^\mu \, , \\ \partial_\mu &\to \Omega \, \partial_\mu \, , \\ \Phi &\to \Omega \, \Phi \, . \end{aligned} \tag{3}$$

This scale invariance of the action is a result of the dimensionless coupling constant λ. The latter property is also essential for the renormalizability of the corresponding perturbation theory. By means of Legendre transformation we derive the Hamiltonian

$$H = \int d^3r \left(\frac{1}{2} \left(\Pi^2 + (\nabla\Phi)^2 \right) + \frac{\lambda}{4!} \, \Phi^4 \right) \, , \tag{4}$$

which is non-negative for $\lambda \geq 0$. In situations, where the Hamiltonian is unbounded from below – e.g. for $\lambda < 0$ – no ground state exists at all. As another example we mention the $\lambda\Phi^3$-theory where the occurrence of arbitrarily negative energies is already present on the classical level. As proven in Ref. [7], this instability persists for the quantized theory. In the case of the $\lambda\Phi^4$-theory the classical ground state is (for $\lambda > 0$) simply given by $\Phi \equiv 0$. Turning to the quantum prescription the situation becomes less clear.

B. The exact quantum vacuum

The main intention of this article is to show that the quantization of the $\lambda\Phi^4$-theory described above is not unique, i.e. it does not maintain all the symmetries of the classical theory. In particular, the vacuum state and thereby the Hilbert space constructed out of it (see the appendix) cannot be scale invariant. In order to prove this assertion, we *assume* that a unique and hence scale invariant vacuum exists and show that this assumption leads to a contradiction. This (fictitious) state is denoted by $|\Psi_\lambda\rangle$ to indicate the dependence on the coupling strength λ.

If we assume that the vacuum would be unique, i.e. scale and Poincaré invariant, it would remain unchanged by the unitary scale transformation $\hat{S}(\Omega)$, which is defined by (see Eq. (3))

$$\hat{S}(\Omega)^{-1} \, \hat{\Phi}(\underline{x}) \, \hat{S}(\Omega) = \Omega \, \hat{\Phi}(\underline{x}/\Omega) \, , \tag{5}$$

i.e. $\hat{S}(\Omega) \, |\Psi_\lambda\rangle = |\Psi_\lambda\rangle$. Otherwise there exists a different vacuum, which can be derived via $\hat{S}(\Omega) \, |\Psi_\lambda\rangle$. Since the scale transformation $\hat{S}(\Omega)$ represents a symmetry of the classical action, the two distinct vacuum states $|\Psi_\lambda\rangle$ and $\hat{S}(\Omega) \, |\Psi_\lambda\rangle$ correspond to two equivalent quantum representations of the classical theory in this situation.

It should be noted here that an anomalous scale dimension (see e.g. [13]) of the fields $\hat{\Phi}$ – inducing a symmetry transformation with other powers in Ω than the one in Eq. (5) – already prevents the theory from being scale-invariant (see also Section II E). But since the proof of this assertion is exactly the aim of this Section we do not assume an anomalous scale dimension *a priori*.

C. Dyson argument

As it is well known, the necessity of introducing a scale already occurs within perturbation theory (renormalization scale). This observation can be interpreted as a hint for the non-uniqueness of the quantization of the $\lambda\Phi^4$-theory. Perturbation theory is a very powerful method that allows the very precise calculation of many observables within quantum field theory for *small* couplings λ, e.g. cross sections, etc. However, properties of the *exact* vacuum state for finite values of the coupling, e.g. $\lambda = 1$, cannot be obtained rigorously within the framework of perturbation theory. The main argument can be traced back to Dyson [8] who applied it to QED; we shall present in the following a modified version of the proof regarding the $\lambda\Phi^4$-theory.

Within perturbation theory one performs a Taylor expansion with respect to the coupling, in our case λ. Especially, the expansion of the exact vacuum $|\Psi(\lambda)\rangle$ would read

$$|\Psi(\lambda)\rangle = \sum_{n=0}^{\infty} \lambda^n \ |\Psi_n\rangle \ . \tag{6}$$

The equal sign above is correct if and only if the infinite summation converges (to the exact quantity). But if this sum converges for some non-vanishing coupling λ_0 then it converges for all (possibly complex) values of λ which satisfy $|\lambda| < \lambda_0$ as well. Accordingly, the expansion above describes an analytic function within the circle of convergence. But in this situation the exact vacuum could be analytically continued to negative values of the coupling λ, where the Hamiltonian is (even classically) unbounded from below. However, in such a highly unstable scenario a (translationally invariant, in particular stationary) vacuum cannot exist. This contradiction leads to the conclusion that the Taylor expansion, i.e. the perturbative approach, does not represent an analytic but an asymptotic expansion.

Therefore, perturbation theory is applicable in a sufficiently small vicinity of the origin $\lambda = 0$ – but not for finite λ such as $\lambda = 1$. Consequently, the fact that the scale invariance is broken in perturbation theory does not necessarily imply that it is broken for the exact vacuum state corresponding to finite λ. Instead one might imagine that the vacuum could be scale invariant at all fixed points, lets say at $\lambda = 1$ and $\lambda = 0$. Within perturbation theory one cannot exclude this possibility – instead one is led to search for non-perturbative methods.

D. Wightman function

Poincaré and scale invariance of the vacuum state impose strong restrictions on the corresponding Wightman functions. Due to the translational symmetry they may depend on the difference of the coordinates $(\underline{x}-\underline{x}')$ only. If we restrict ourselves to a region away from the light-cone $(\underline{x}-\underline{x}')^2 \neq 0$ Lorentz invariance implies, that merely the scalar $(\underline{x}-\underline{x}')^2$ enters the Wightman function. Taking into account the scale invariance $W(\Omega^{-1}\underline{x}, \Omega^{-1}\underline{x}') = \Omega^2 W(\underline{x}, \underline{x}')$ the two-point function has to adopt the following form

$$W(\underline{x}, \underline{x}') = \frac{\text{const}}{(\underline{x}-\underline{x}')^2} \tag{7}$$

for $(\underline{x}-\underline{x}')^2 \neq 0$. By inspection we observe that the action of the d'Alembert operator $\Box = \partial_\mu \partial^\mu$ on this function yields zero. At first this holds away from the light cone.

To examine additional contributions on the light cone such as $\delta[(\underline{x}-\underline{x}')^2]$ we investigate the Fourier transform $\widetilde{W}$. Every positive $L_+^\uparrow$-invariant distribution $\widetilde{\zeta}$ with support in the closed forward cone $\mathrm{supp}(\widetilde{\zeta}) \subseteq \overline{V_+}$ has to take the form (see [9,3] and [6], Theorem IX.33)

$$\widetilde{\zeta}(\underline{k}) = a\, \delta^4(\underline{k}) + \Theta(k_0)\, \mu(\underline{k}^2) \tag{8}$$

with $a \geq 0$ and a positive measure $\mu \geq 0$ with $\mathrm{supp}(\mu) \subseteq \overline{\mathbb{R}_+}$. In view of the Wightman axioms the Fourier transform of the Wightman function $\widetilde{W}(\underline{k})$ has to be represented by a special choice of $\widetilde{\zeta}(\underline{k})$. The above theorem allows for the Källén-Lehmann spectral representation [10] of the Wightman function

$$W(\underline{x}, \underline{x}') = a + \int d\mu(m^2)\, W^{\text{free}}(\underline{x}, \underline{x}', m^2) \tag{9}$$

where $W^{\text{free}}(\underline{x}, \underline{x}', m^2)$ denotes the Wightman function of a free scalar field with mass m, cf. [3] and [6], Theorem IX.34. The imposed scale invariance of the Wightman function $\widetilde{W}(\Omega^2 \underline{k}^2) = \widetilde{W}(\underline{k}^2)/\Omega^2$ implies $a = 0$ and $\mu(\Omega^2\chi) = \mu(\chi)/\Omega^2$. As a consequence, if μ contributes for positive χ then it has to behave (for $\chi > 0$) like $\mu(\chi) = b/\chi$ with $b \geq 0$. However, the resulting quantity $\widetilde{\zeta}(\underline{k}) = \Theta(k_0)\,\Theta(\underline{k}^2) b/\underline{k}^2$ does not represent a well-defined distribution owing to the singularity at $\underline{k}^2 = 0$ together with the Heaviside step-function Θ. Equivalently it does not possess a Fourier transform. This can easily be verified by considering

$$\Box\zeta(\underline{x}, \underline{x}') = -b\, \mathcal{F}\left(\Theta(k_0)\Theta(\underline{k}^2)\right) = \frac{-8\pi b}{(\underline{x}-\underline{x}')^4} \tag{10}$$

which yields for $(\underline{x}-\underline{x}')^2 > 0$. But no scale invariant distribution exists which generates the r.h.s. of the above equation when the d'Alembert operator is applied to. On the contrary – as we have observed in Eq. (7) – the action of the d'Alembert operator on the Wightman function yields zero – at least for $(\underline{x}-\underline{x}')^2 \neq 0$. As a result, the support of the measure μ can merely contain the point $\chi = 0$. There exists only one positive distribution with support at the origin – the Dirac δ-function. Ergo, the remaining possibility for the Fourier transform of the Wightman function is given by

$$\widetilde{W}(\underline{k}) = \Theta(k_0)\, \delta(\underline{k}^2) \cdot \text{const}\,. \tag{11}$$

This quantity indeed obeys scale invariance. In conclusion assuming a unique vacuum the d'Alembert operator acting on the Wightman function vanishes everywhere

$$\underline{k}^2\, \widetilde{W}(\underline{k}) = 0 \;\leftrightarrow\; \Box\, W(\underline{x}, \underline{x}') = 0\,. \tag{12}$$

and in particular on the light cone.

E. Equation of motion

The variation of the action $\delta\mathcal{A} = 0$ leads to the non-linear equation

$$\Box\hat{\Phi} = -\frac{\lambda}{3!}\, \hat{\Phi}^3\,. \tag{13}$$

The field $\hat{\Phi}(\underline{x})$ is represented by an operator-valued distribution. However, the product of two or more distributions with the same argument, for example $[\delta(x)]^3$ is not well-defined in general. Consequently, the source term on the r.h.s. of the equation above $[\hat{\Phi}(\underline{x})]^3$ has at first glance no definite meaning. Strictly speaking, we have to define the non-linear source term $\hat{j} = \Box\hat{\Phi} = -\lambda\hat{\Phi}^3/3!$ as a local operator-valued tempered distribution. By virtue of the equation of motion it has to obey the following relation under rescaling, see Eq. (5)

$$\hat{S}(\Omega)^{-1}\, \hat{j}(\underline{x})\, \hat{S}(\Omega) = \Omega^3\, \hat{j}(\underline{x}/\Omega)\,. \tag{14}$$

As already discussed in Section II B, the occurrence of an anomalous scale dimension of the fields $\hat{\Phi}(\underline{x})$ or – more generally – the introduction of a renormalization scale Λ_{R} in order to define $\hat{j}$, i.e. $\hat{j} = \hat{j}(\Lambda_{\mathrm{R}})$, violate this condition. But in this case the proof of the non-uniqueness of the exact vacuum state is already complete at this stage: In this situation the vacuum has to depend on this renormalization scale as well. Otherwise the difference of two source terms corresponding to different scales acting on the vacuum (supposed to be invariant) yields zero

$$\left(\hat{j}(\Lambda_{\mathrm{R}}) - \hat{j}(\Lambda'_{\mathrm{R}})\right) |\Psi_\lambda\rangle = 0 \tag{15}$$

according to the equation of motion (13). With the same arguments as used at the end of the next Section this implies $\hat{j}(\Lambda_{\mathrm{R}}) - \hat{j}(\Lambda'_{\mathrm{R}}) = 0$, which contradicts the assumption of a scale dependent source. In summary, the eventual necessity of introducing a renormalization scale in order to define $\hat{j}$ would result in a dependence of the vacuum on this scale.

F. Federbush-Johnson theorem

As shown in Sec. II D assuming the existence of a unique vacuum the two-point Wightman function equals (up to an irrelevant pre-factor) the two-point function of the free field. On the other hand, we may now exploit the following trivialization theorem, which is sometimes called the Federbush-Johnson theorem: *If the two-point function coincides with its free-field analogue then the theory is free*, see [5,11–14].

In the following we sketch a proof of this theorem: If the action of the d'Alembert operator on the two-point Wightman function yields zero we may utilize the equation of motion via

$$\begin{aligned}\Box\Box' W(\underline{x},\underline{x}') &= \langle\Psi_\lambda|\,\Box\hat{\Phi}(\underline{x})\Box'\hat{\Phi}(\underline{x}')\,|\Psi_\lambda\rangle \\ &= \left(\frac{\lambda}{3!}\right)^2 \langle\Psi_\lambda|\,\hat{\Phi}^3(\underline{x})\hat{\Phi}^3(\underline{x}')\,|\Psi_\lambda\rangle \\ &= 0\,. \end{aligned} \tag{16}$$

This equality holds for all $\underline{x}$ and $\underline{x}'$ and especially for $\underline{x} = \underline{x}'$. Accordingly, we obtain $\langle\Psi_\lambda|\,[\hat{\Phi}^3(\underline{x})]^2\,|\Psi_\lambda\rangle = 0$, which implies $\hat{\Phi}^3(\underline{x})\,|\Psi_\lambda\rangle = 0$. The last conclusion was possible because the Hilbert space $\mathfrak{H}$ possesses a positive definite scalar product, for a Fock space with an indefinite metric additional considerations are necessary.

Now we may exploit the postulated cyclicity $\mathfrak{A}\,|\Psi_\lambda\rangle = \mathfrak{H}$ of the vacuum (see the appendix). This property implies that all states of the Hilbert space can be approximated by polynomials of fields (smeared with test functions) acting on the vacuum. Utilizing analyticity arguments (theorem of identity for holomorphic functions) it can be shown that it is sufficient to employ test functions with support in an arbitrary small open domain $\mathcal{O}$. This fact is known as Reeh-Schlieder [15] theorem: $\overline{\mathfrak{A}(\mathcal{O})\,|\Psi_\lambda\rangle} = \mathfrak{H}$. One consequence of this theorem is the fact that if a local operator annihilates the vacuum, it is the zero operator, cf. [3] and [5]. As a result the annihilation of the vacuum $\hat{\Phi}^3(\underline{x})\,|\Psi_\lambda\rangle = 0$ again implies $\hat{\Phi}^3 = 0$, i.e. a free theory.

In a similar way one can also show that the field does not only satisfy the equation of motion but also the commutation relations of a free field [16]. This can be demonstrated via considering the quantity

$$\hat{\mathfrak{G}}(\underline{x},\underline{x}') = \left[\hat{\Phi}(\underline{x}),\hat{\Phi}(\underline{x}')\right] - \langle\Psi_\lambda|\left[\hat{\Phi}(\underline{x}),\hat{\Phi}(\underline{x}')\right]|\Psi_\lambda\rangle \tag{17}$$

and an argumentation similar to the one above, see [13] and [14].

The proof by Federbush and Johnson in Ref. [11] employs canonical commutation relations and analyticity arguments but it does not refer to the Reeh-Schlieder property, which was established later.

A completely different argument indicating the unphysical consequences of the annihilation of the vacuum by the source term is based on the natural assumption that the free theory should be recovered in the limit $\lambda \to 0$. Hence, the independence of the identity $\hat{\Phi}^3(\underline{x})\,|\Psi_\lambda\rangle = 0$ of the coupling λ is in conflict to the fact, that $\hat{\Phi}^3\,|0\rangle = 0$ is not valid in the non-interacting situation.

In summary, these contradictions lead to two alternatives: Either the self-interaction vanishes or the vacuum is not unique. In the former case the vacuum is Poincaré and scale invariant, but the theory is trivial, see e.g. [17]. Assuming a non-trivial self-interacting $\lambda\Phi^4$-theory (latter case) *no* unique vacuum exists.

III. SYMMETRY BREAKING

As we have shown in the previous Section a regular state that obeys all symmetries of the considered theory does not exist. Accordingly, the only possibility to define a vacuum is to break at least one of the symmetries. Certainly one agrees that the Poincaré invariance in fundamental field theories on a Minkowski space-time should not be broken. Without this symmetry it is by no means obvious how to distinguish the vacuum from all other states. As a consequence, we have to break the only symmetry left, i.e. the scale invariance. Even though the action exhibits no specific scale, the introduced vacuum now displays a scale-dependence. We denote the scale of the symmetry breaking by Λ_Φ and the vacuum accordingly by $\left|\Psi_\lambda^\Lambda\right\rangle$.

In the following we are going to analyze the consequences of this *Ansatz*. To investigate the relation of the vacuum state $\left|\Psi_\lambda^\Lambda\right\rangle$ to the ground state of the theory we have to evaluate the renormalized expectation value of the energy density, i.e. the 00-component of the energy-momentum tensor. In conjunction with the Wightman formalism it is most convenient to employ the powerful point-splitting renormalization technique [18], which is well-established in quantum field theory on curved space-times. With this tool we are able to calculate the expectation value of the energy-momentum tensor and the phionic and scalar condensates.

A. Point-splitting

Several interesting observables, e.g. the energy-momentum tensor, contain two or more fields at equal space-time points $\hat{A}(\underline{x})\hat{B}(\underline{x})$. Due to the singular character of the product of two distributions with the same argument such quantities usually diverge $\langle\hat{A}(\underline{x})\hat{B}(\underline{x})\rangle = \infty$. This necessitates an appropriate regularization and renormalization scheme. Having at hand merely the Wightman functions as input information the only well-known procedure, which can be applied directly, is the point-splitting method.

Accordingly, one at first considers the fields at distinct space-time points $\langle\hat{A}(\underline{x}')\hat{B}(\underline{x})\rangle < \infty$ and takes the coincidence limit afterwards – a method called point-splitting

regularization. In order to generate physical reasonable, i.e. finite (renormalized) results, those terms, which become singular in the limit $\underline{x}' \to \underline{x}$, have to be discarded.

The physical meaning of the renormalization scheme described above can be understood by considering a physical reasonable measurement process. Realistic detectors always produce finite results. Due to the fact, that those detectors are not point-like, but exhibit a finite extension the corresponding expectation values are finite as well. A linear detector (in the free field example a one-particle detector) can be described by the product of two fields smeared with the test functions F and G, see Eq. (39). The response of that detector is given by the (finite) expectation value $\langle\hat{\Phi}(F)\,\hat{\Phi}(G)\rangle$. In order to consider a divergent expectation value – for instance $\langle\hat{\Phi}^2\rangle$ – as a limiting case of responses of suitable detectors one may proceed as follows: At first the space-time supports of the test functions F and G shrink to non-coinciding points $\langle\hat{\Phi}(\underline{x})\hat{\Phi}(\underline{x}')\rangle$. The associated expectation value is still finite. Then one considers the coincidence limit $\underline{x} \to \underline{x}'$, where the expectation value diverges. Accordingly, this idealization of a physical detector exactly corresponds to the point-splitting procedure.

The mechanism described above can be elucidated by a simple example. Let us consider the following *Ansatz* for the exact two-point Wightman function:

$$W(\underline{x}, \underline{x} + \underline{\Delta x}) = \sum_n a_n(\Lambda_\mathrm{F})\,\underline{\Delta x}^{2n}\,\Lambda_\mathrm{F}^{2n+2} + \sum_n b_n(\Lambda_\mathrm{F})\,\underline{\Delta x}^{2n}\,\Lambda_\mathrm{F}^{2n+2}\,\ln\left|\underline{\Delta x}^2\,\Lambda_\mathrm{F}^2\right|. \tag{18}$$

(For reasons of simplicity we restrict ourselves to space-like separations.) This expansion is correct for a sufficiently well-behaving spectral measure $\mu(m^2)$ in the Källén-Lehmann representation in Eq. (9). For more complicated measures further terms such as $\ln^2\left|\underline{\Delta x}^2\,\Lambda_\mathrm{F}^2\right|$ may appear and create additional singularities without altering the following considerations. Owing to the occurrence of the logarithmic terms we had to introduce a scale Λ_F. As a variation of this scale induces a transfer of contributions from the b_n to the a_n terms these coefficients may dependent explicitly on Λ_F. This scale characterizes the way of distinguishing the different contributions to the response of the detector. For the moment it is completely determined by the observer and should not be confused with the scale of symmetry breaking Λ_Φ, which is a property of the vacuum.

Having at hand the explicit expression for the Wightman function we are now able to derive the renormalized expectation value of $\hat{\Phi}^2$. To this end one keeps only those terms of the above equation, which are finite in the coincidence limit, i.e. one obtains

$$\langle\hat{\Phi}^2\rangle_\mathrm{ren}^{\Lambda_\mathrm{F}} = a_0(\Lambda_\mathrm{F})\,. \tag{19}$$

Note, that the renormalized expectation values may depend on the the scale Λ_F as well.

B. Operator product expansion

In order to elucidate the physical meaning of the introduced scales Λ_F and Λ_Φ we examine their relation to an important and powerful tool in quantum field theory – the operator product expansion [19]. Considering the expectation value of a product of two operators at distinct space-time points as non-local quantity it is possible to perform an expansion into a sum of local operators with non-local coefficients

$$\left\langle\hat{A}(\underline{x} + \underline{\Delta x}/2)\hat{B}(\underline{x} - \underline{\Delta x}/2)\right\rangle = \sum_n C_n(\underline{\Delta x}, \Lambda_\mathrm{F})\left\langle\hat{O}_n(\underline{x}, \Lambda_\mathrm{F})\right\rangle. \tag{20}$$

Within the framework of the operator product expansion Λ_F is denoted as the factorization scale. Considering the most simple example $\hat{A} = \hat{B} = \hat{\Phi}$ leads us back to the Wightman function in Eq. (18). The operator corresponding to $(\underline{\Delta x}^2)^{n=0} = \mathrm{const}$ exactly represents the second-order scalar condensate $\langle\hat{O}_0\rangle = \langle\hat{\Phi}^2\rangle_\mathrm{ren}$. Calculating the expectation values of the local operators $\hat{O}_n$ in the vacuum $\left|\Psi_\lambda^\Lambda\right\rangle$, which possesses the symmetry breaking scale Λ_Φ, this scale obviously enters the local quantities $\langle\hat{O}_n\rangle$ as well. By inspection we observe that for small distances $\underline{\Delta x}^2\Lambda_\Phi^2 \ll 1$ the lowest-order term is most relevant. This contribution describes the short range behavior of the theory. On the other hand for large distances $\underline{\Delta x}^2\Lambda_\Phi^2 = \mathcal{O}(1)$ higher-order contributions become more and more relevant. Accordingly, the long range features – mediated via the operators $\hat{O}_n$ – usually dominate in this situation. Evidently, the symmetry breaking scale can be envisaged as the *natural scale of factorization*, which distinguishes between the long and short range behavior $\Lambda_\Phi = \Lambda_\mathrm{F}$.

C. Observables

In analogy to the second-order scalar condensate $\langle\hat{\Phi}^2\rangle_\mathrm{ren} = a_0$ we may derive further renormalized expectation values by means of an appropriately chosen differential operator acting on the Wightman function in Eq. (18). The contributions which are finite in the coincidence limit read

$$\langle\hat{\Phi}\Box\hat{\Phi}\rangle_\mathrm{ren} = 4\Lambda_\Phi^4(2a_1 + 3b_1)\,. \tag{21}$$

By virtue of Poincaré invariance we can deduce

$$\langle\partial_\mu\hat{\Phi}\partial_\nu\hat{\Phi}\rangle_\mathrm{ren} = -g_{\mu\nu}\Lambda_\Phi^4(2a_1 + 3b_1)\,. \tag{22}$$

Utilizing the equation of motion (13) we derive from Eq. (21) the fourth-order scalar condensate

$$\langle\hat{\Phi}^4\rangle_\mathrm{ren} = -\frac{4!}{\lambda}\Lambda_\Phi^4(2a_1 + 3b_1)\,, \tag{23}$$

where we have used the definition $\hat{\Phi}^4 = \hat{\Phi}^3\hat{\Phi}$ that is consistent with the equation of motion and the Heisenberg representation. The factor $1/\lambda$ indicates that our non-perturbative results cannot be obtained using elementary perturbation theory. The expectation value of the Lagrangian density corresponding to the phionic condensate yields

$$\langle\hat{\mathcal{L}}\rangle_{\rm ren} = -\Lambda_\Phi^4(2a_1 + 3b_1)\,. \tag{24}$$

These ingredients enable us to achieve one of our main intentions – the derivation of the energy-momentum tensor $T_{\mu\nu} = \partial_\mu\Phi\partial_\nu\Phi - g_{\mu\nu}\,\mathcal{L}$. We observe the exact cancellation of the above contributions

$$\langle\hat{T}_{\mu\nu}\rangle_{\rm ren} = 0\,. \tag{25}$$

Note, that for deriving this equation we merely need Poincaré invariance (together with the point-splitting technique). As a counter-example one may remember the Casimir effect where $\langle\hat{T}_{00}\rangle_{\rm ren} < 0$ holds, even for $\lambda = 0$.

The above result indeed confirms the identification of the vacuum state $\left|\Psi_\lambda^\Lambda\right\rangle$ with a ground state. Because of $\hat{H} = \int d^3r\,\hat{T}_{00}$ it follows $\langle\hat{H}\rangle_{\rm ren} = 0$ from the equation above. (The spectral condition explained in the appendix together with the scale invariance implies $\hat{H}_{\rm ren} \geq 0$.) Ergo, any of the introduced vacua $\left|\Psi_\lambda^\Lambda\right\rangle$ characterized by a positive value of the scale Λ_Φ may be identified as ground states of the theory (which are therefore not unique).

It should be emphasized, that the zero expectation value in Eq. (25) is rather non-trivial. Considering a massless scalar field with a $\lambda\Phi^n$-coupling in a D-dimensional space-time one arrives at

$$\langle T_{\mu\nu}\rangle_{\rm ren} \sim g_{\mu\nu}\Lambda_\Phi^n\left(1 - \frac{D}{2} + \frac{D}{n}\right)\,. \tag{26}$$

Cancellations similar to the situation discussed above occur exactly in those cases, where the theory is scale invariant, i.e. for $2(n + D) = nD$. In addition to the absence of any mass terms the scale invariance implies a dimensionless coupling constant.

IV. CONCLUSIONS

A. Summary

Utilizing the Wightman axioms we have shown for the scalar, massless, neutral, and self-interacting $\lambda\Phi^4$-theory that no state exists, which preserves Poincaré as well as scale invariance, i.e., all the symmetries of the Lagrangian. Accordingly, we are lead to introduce the non-perturbative vacuum state by breaking the scale invariance. Consistent with the Wightman approach we employed the point-splitting technique, which allows for an explicit evaluation of renormalized expectation values. Within this formalism we calculated the scalar as well as the phionic condensate. The renormalized vacuum expectation value of the energy-momentum tensor vanishes, which implies that all vacua $\left|\Psi_\lambda^\Lambda\right\rangle$ are ground states.

B. Discussion

Raising the question about the existence of a unique vacuum state in a field theory including a non-trivial interaction term we focused on the real and massless $\lambda\Phi^4$-theory. Perturbation theory is based on the vacuum of the free theory, which can be uniquely determined and coincides with the ground state of the corresponding free Hamiltonian. It is evident from the beginning that a unique vacuum state should respect all the symmetries of the underlying theory, i.e. that of the Lagrangian. Since the vacuum state is defined via the free theory in perturbation theory, this property of the vacuum is established by brute force and is independent of the form of the interaction term in the Lagrangian.

In the framework of non-perturbative methods there is a need for the definition of a corresponding exact vacuum state of the interacting theory. Naively this state should again respect all the symmetries of the underlying Lagrangian but now incorporating the interaction term. The latter will in general have some impact on the vacuum state. As indicated below the structure of the exact vacuum state becomes rather complex in comparison with the free (perturbative) vacuum.

As a generic example we checked, whether such a non-perturbative vacuum state can be found in the $\lambda\Phi^4$-theory. To this end we started with the definition of a vacuum state as a state, which preserves all the symmetries of the Lagrangian: Poincaré and scale invariance. However, it turned out that the conjectured vacuum state does not allow for the generation of all other states in the self-interacting theory by means of field operators, and thus it is inconsistent with the property of cyclicity of vacuum states. Therefore the vacuum cannot be unique.

We conclude that the only reasonable way to define a vacuum of $\lambda\Phi^4$-theory is to break scale invariance. As a consequence, the vacuum state of the theory now depends on a new scale Λ_Φ. We were able to interpret this scale in the framework of operator product expansion (OPE), where the expectation value of a field product is decomposed into a sum of products consisting of two parts describing the long range and short range behavior, respectively. Here the scale Λ_Φ is to be identified with the factorization scale of OPE, i.e. with the scale separating the components of long and short distances. This is of considerable importance in theories with asymptotic freedom for which the short distance dependent part may be calculated perturbatively. However, this is not the case for the $\lambda\Phi^4$-theory because of its QED-like asymptotic

behavior (Landau pole).

Finally we were investigating possible consequences of the new scale Λ_Φ and its appearance in observable quantities of the $\lambda\Phi^4$-theory. Starting from a very general structure for the two-point Wightman function, we found that the expectation value of the energy-momentum tensor vanishes non-trivially. *Non-trivially* means an exact cancellation of the scale dependence in both terms contributing to the energy momentum tensor which occurs for scale invariant Lagrangians only. This zero result confirms the notion of the scale dependent vacuum state as a ground state of the theory.

This result may also be compared to corresponding results obtained in the framework of perturbation theory and one may ask about the relation of our zero result to the known trace anomalies [20,21]. In Ref. [21] the following expression for the trace anomaly has been derived

$$\langle T^\mu_\mu\rangle_{\mathrm{ren}} = -\frac{\beta}{4!}\,\langle\Phi^4\rangle_{\mathrm{ren}}\,. \tag{27}$$

One should be aware that within renormalization theory perturbative results keep the same form independently of the momentum flow through the corresponding Feynman diagrams. Actually, in this paper we calculated the *vacuum* expectation value of the energy-momentum tensor. Owing to the translation invariance of the vacuum the Fourier transform of every local expectation value contributes only at vanishing momentum. In order to compare our result in Eq. (25) with Eq. (27) we have to evaluate the quantities there – especially the β-function – at zero momentum. Since the $\lambda\Phi^4$-theory obeys an infrared fixed point ($\beta = 0$) our zero-result $\langle T^\mu_\mu\rangle_{\mathrm{ren}} = 0$ for the energy-momentum tensor is in accordance with the calculations within perturbation theory – even though $\langle\Phi^4\rangle_{\mathrm{ren}} \neq 0$. Nevertheless one should be careful in comparing perturbative and non-perturbative results, as one cannot expect in general that a non-perturbative result has a relation to any finite order perturbative calculation. In addition the comparison of results obtained within different renormalization procedures (i.e. dimensional and point-splitting) is a delicate task.

To elucidate the properties and the complex nature of the non-perturbative vacuum, we may analyze this state by considering e.g. its content of free particles $\hat{N}^{\mathrm{free}}_{\vec{k}}$. Applying this number operator to the free vacuum yields zero and it diagonalizes the free Hamiltonian $\hat{H}(\lambda = 0) = \hat{H}^{\mathrm{free}} = \int d^3r\,\hat{T}^{\mathrm{free}}_{00}$. The simultaneous ground state of all these non-negative operators $\hat{N}^{\mathrm{free}}_{\vec{k}}$ is unique and coincides with the free vacuum. To calculate their expectation values it is sufficient to know the two-point function. Owing to the deviation of the exact Wightman function of the interacting theory from the free (scale invariant) two-point function (as proved in Section II) at least one expectation value differs from zero

$$\left\langle\Psi^\Lambda_\lambda\right|\hat{N}^{\mathrm{free}}_{\vec{k}}\left|\Psi^\Lambda_\lambda\right\rangle > 0 \tag{28}$$

indicating that the non-perturbative vacuum contains a non-vanishing amount of "free" scalar particles. This provides another hint for the non-triviality of the zero result in Eq. (25). The non-perturbative vacuum contains exactly such an amount of free particles that the contributions to the energy-momentum tensor of the interacting theory cancel.

Traditional scattering theory is based on asymptotically free particles in the in- and out-states. For energy ranges where $\left\langle\Psi^\Lambda_\lambda\right|\hat{N}^{\mathrm{free}}_{\vec{k}}\left|\Psi^\Lambda_\lambda\right\rangle$ yields significant contributions the naive application of the above formalism is not obviously justified. Instead the propagation of the particles is similar to that in a medium.

The necessity of breaking the scale symmetry in a non-perturbative approach has consequences to the application of the path-integral formalism. The generating functional

$$W[J] = \int \mathcal{D}\Phi\,\exp\left(i\int d^4x\,\mathcal{L} + J\Phi\right)\,, \tag{29}$$

if it exists beyond perturbation theory with the usual regular measure $\mathcal{D}\Phi$, is scale invariant per definition. So are all expectation values deduced of it. Usually these expectation values may be identified with the vacuum expectation values, which are then scale invariant as well. But this is inconsistent with the scale dependence of the exact vacuum state. It follows that the usual scale invariant path-integral formalism is not naively applicable to non-perturbative analytical calculations in the case of the $\lambda\Phi^4$-theory. Of course, the argument presented above does not apply to lattice calculations where the lattice spacing induces a scale which may be connected with the intrinsic scale of the vacuum.

In summary the results obtained so far motivate further examinations concerning the relation of the presented non-perturbative approach to other formalisms. Furthermore, it might be interesting to extend the method for the explicit non-perturbative evaluation of expectation values – as presented in this article – to other observables.

C. Outline

We expect that the non-uniqueness of the vacuum state is a more general feature, which holds true in other scale invariant field theories as well. This may especially be the case for the gauge sector of QCD – a statement which is currently under consideration [22]. If so, our assertion may have consequences for the current efforts to find a treatable approach in QCD in the medium energy range of some GeV.

The Lagrangian governing the dynamics of the gluons $\mathcal{L} = -G^a_{\mu\nu}G^{\mu\nu}_a/4$ is scale invariant as the Lagrangian of the $\lambda\Phi^4$-theory. In contrast to the latter case further difficulties arise. The character of this field theory as a

gauge field theory implies primary and secondary constraints. The equations of motion are more involved and contain terms linear and quadratic in the coupling g. On the other hand there is an additional $SU(3)$-color symmetry.

In QCD the expectation value of the Lagrangian density of the gluonic sector $\langle \hat{G}^a_{\mu\nu} \hat{G}^{\mu\nu}_a \rangle_{\rm ren}$ represents the gluonic condensate. The calculation of this quantity in analogy to Sec. III C might provide some interesting insights owing to its considerable relevance in QCD sum rules (see e.g. [23]) and more generally for OPE.

The energy-momentum tensor of the pure gluonic sector is traceless at the classical level. Then Poincaré invariance would imply the vanishing of its renormalized vacuum expectation value

$$\langle \hat{T}_{\mu\nu} \rangle_{\rm ren} = 0 \quad . \tag{30}$$

Nevertheless, in analogy to the $\lambda\Phi^4$-theory [20,21] the phenomenon of a trace anomaly occurs in QCD as well [24]. Since the Yang-Mills theory possesses a low momentum behavior, which is different from the $\lambda\Phi^4$-theory the arguments presented in Sec. IV B do not necessarily apply in this case. This may result in a non-vanishing expectation value.

V. APPENDIX: THE WIGHTMAN AXIOMS

For the free field there exist two different options to define the vacuum, firstly as the ground state of the Hamiltonian and secondly as the state which is Poincaré invariant. For the interacting field the former possibility does not apply in general. In the following we recapitulate an axiomatic approach to quantum field theory based on the Wightman [4] formalism that utilizes Poincaré invariance. The quantum field $\hat{\Phi}$ is represented by an operator-valued tempered distribution acting on a separable Hilbert space $\mathfrak{H}$. The convolution of operator-valued tempered distributions with smooth test functions of compact support yields regular operators which generate an algebra $\mathfrak{A}$. Poincaré transformations are mediated via unitary operators $\hat{U}(\underline{L}, \underline{a})$

$$\hat{U}(\underline{L}, \underline{a})^{-1}\, \hat{\Phi}(\underline{x})\, \hat{U}(\underline{L}, \underline{a}) = \hat{\Phi}(\underline{L}\underline{x} + \underline{a})\,. \tag{31}$$

The Hilbert space $\mathfrak{H}$ possesses a cyclic and Poincaré invariant $\hat{U}(\underline{L}, \underline{a})\, |\Psi_0\rangle = |\Psi_0\rangle$ state $|\Psi_0\rangle$ which is called the vacuum. Per definition of cyclicity, all other states $|\Psi\rangle$ of the Hilbert space $\mathfrak{H}$ can be created by acting an appropriate functional $F_\Psi[\hat{\Phi}]$ on the vacuum

$$\begin{aligned} |\Psi\rangle &= F_\Psi[\hat{\Phi}]\, |\Psi_0\rangle \ , \\ \mathfrak{H} &= \mathfrak{A}\, |\Psi_0\rangle \ . \end{aligned} \tag{32}$$

As a consequence, the expectation values of all observables in all states can be expressed in terms of vacuum expectation values of field operators – the Wightman functions (reconstruction theorem). In order to represent a realistic field theory the Wightman functions have to fulfill certain properties. These axioms are presented in the following for the example of the two-point function for a neutral scalar field $\hat{\Phi}$, see e.g. [3–6].

A. Definition

To ensure the character of the quantum field as an operator-valued tempered distribution the two-point Wightman function

$$W(\underline{x}, \underline{x}') = \langle \Psi_0 |\, \hat{\Phi}(\underline{x}) \hat{\Phi}(\underline{x}')\, | \Psi_0 \rangle \tag{33}$$

has to be a tempered bi-distribution. The property of neutral fields to be described by hermitian operators implies

$$W^*(\underline{x}, \underline{x}') = W(\underline{x}', \underline{x})\,. \tag{34}$$

B. Covariance

In order to generate a Poincaré invariant vacuum the Wightman functions must exhibit the same feature

$$W(\underline{x}, \underline{x}') = W(\underline{L}\underline{x} + \underline{a}, \underline{L}\underline{x}' + \underline{a}) \tag{35}$$

for all translations $\underline{a}$ and all rotations $\underline{L}$ of the restricted Lorentz group $L^\uparrow_+ = \{\underline{L} : \det\underline{L} = 1, L^0_0 > 0\}$, which contains all transformations connected continuously to the identity, i.e. no time and/or space inversion. Translation invariance implies the Wightman function to depend on the difference of the coordinates $\underline{x} - \underline{x}'$ only. Inside of each light cone merely $(\underline{x} - \underline{x}')^2$ enters the Wightman functions due to rotational symmetry. However, they may differ in their values inside the future and the past light cone, respectively.

C. Spectral condition

The properties listed above allow for the Fourier transformation of the Wightman function according to

$$\begin{aligned} W(\underline{x}, \underline{x}') &= \mathcal{F}\left(\widetilde{W}\right) \\ &= \int d^4k\, \widetilde{W}(\underline{k})\, \exp\left(-i\underline{k}(\underline{x} - \underline{x}')\right)\,. \end{aligned} \tag{36}$$

It should be stated that all considerations employ the Minkowski metric $g_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$. To ensure the stability of the theory the support of this Fourier transform $\widetilde{W}(\underline{k})$ has to be contained in the closed forward cone $\overline{V_+} = \{\underline{k} : \underline{k}^2 \geq 0, k_0 \geq 0\}$

$$k_0 < 0 \to \widetilde{W}(\underline{k}) = 0$$
$$\underline{k}^2 < 0 \to \widetilde{W}(\underline{k}) = 0\,. \qquad (37)$$

k_0 is related to the energy and thus, the first condition $k_0 \geq 0$ prevents the system from collapsing. Poincaré symmetry implies the vanishing of the Fourier transform in the whole space-like region.

D. Locality

By means of Einstein causality space-like separated events cannot interfere. As a result we require the fields to commute at space-like distances and therefore the Wightman functions to be symmetric in this case

$$(\underline{x} - \underline{x}')^2 < 0 \;\to\; W(\underline{x}, \underline{x}') = W(\underline{x}', \underline{x})\,. \qquad (38)$$

For neutral fields the Wightman functions are therefore completely real at space-like distances.

E. Positivity

Smearing the (hermitian) operator-valued distributions $\hat{\Phi}(\underline{x})$ with smooth test functions of compact support $G(\underline{x})$ one acquires regular operators

$$\hat{\Phi}(G) = \int d^4x\, \hat{\Phi}(\underline{x})\, G(\underline{x})\,. \qquad (39)$$

The absolute value squared of an operator $|\hat{\Phi}(G)|^2 = [\hat{\Phi}(G)]^\dagger \hat{\Phi}(G) = \hat{\Phi}(G^*)\hat{\Phi}(G)$ and thereby also its expectation value are non-negative. Therefore the Wightman functions have to obey the following positivity condition for all test functions G

$$\int d^4x \int d^4x'\, G^*(\underline{x})\, W(\underline{x}, \underline{x}')\, G(\underline{x}') \geq 0\,. \qquad (40)$$

Applying the Fourier transformation on this inequality the positivity requirement takes the very simple form in terms of the Fourier transform of the Wightman function

$$\widetilde{W}(\underline{k}) \geq 0\,. \qquad (41)$$

F. Cluster property

The existence of a unique translationally invariant state (i.e. the vacuum) $|\Psi_0\rangle$ is used (cf. [3]) to deduce the cluster property of quantum field theories

$$\lim_{\underline{s}^2 \to -\infty} \langle \Psi_0|\; \hat{A}(\underline{x} + \underline{s})\, \hat{B}(\underline{x}')\; |\Psi_0\rangle$$
$$= \langle \Psi_0|\; \hat{A}(\underline{x})\; |\Psi_0\rangle\; \langle \Psi_0|\; \hat{B}(\underline{x}')\; |\Psi_0\rangle\;, \qquad (42)$$

where $\hat{A}, \hat{B}$ are operators composed out of fields. This property is crucial for defining the S-matrix [3]. The existence of more than one translationally invariant state in the Hilbert space $\mathfrak{H}$ would imply that the cluster property does not hold in general. However, for operators associated with physically meaningful events the cluster property should remain valid, because events at large space-like distances are asymptotically uncorrelated. Recalling the scale-dependence and thereby non-uniqueness of the vacuum of the considered $\lambda\Phi^4$-theory one is lead to the question whether the cluster property is satisfied in this case. Since the Hilbert space is constructed starting from the cyclic vacuum $\mathfrak{H}(\lambda, \Lambda_\Phi) = \mathfrak{A}\left|\Psi_\lambda^\Lambda\right\rangle$ it may also depend on the scale. The remaining question is, whether different vacuum states corresponding to different scales belong to the same Hilbert space or not, i.e. whether

$$\left|\Psi_\lambda^\Lambda\right\rangle \in \mathfrak{H}(\lambda, \Lambda'_\Phi)\,. \qquad (43)$$

Indeed, it is possible that different values of the scale Λ_Φ correspond to distinct Hilbert spaces $\mathfrak{H}(\lambda, \Lambda'_\Phi)$, which are not connected by local excitations. Such a situation, where different global features generate distinct Hilbert space representations (super-selection sectors), occurs for example in field theories at different values of the temperature. If the physically realized vacuum state coincides with such a vacuum state – which generates a Hilbert space containing only one translationally invariant state – then the cluster property still holds.

Acknowledgment

The authors are grateful to H. Araki, D. Diakonov, R. Jackiw, F. Krauss and D. J. Schwarz for valuable conversations and kind comments. R. S., R. K., and G. P. acknowledge fruitful discussions with K. Sailer and Z. Schram during their stay at the Department of Theoretical Physics at the University of Debrecen, Hungary. This visit was supported by MÖB and DAAD. Financial support from BMBF, DFG and GSI is gratefully acknowledged.

---

[1] R. Jackiw and C. Rebbi, Phys. Rev. Lett. **37**, 172 (1976).

[2] E. D'Hocker and R. Jackiw, Phys. Rev. D **26**, 3517 (1982). E. D'Hocker, D. Z. Freedman, and R. Jackiw, Phys. Rev. D **28**, 2583 (1983). E. D'Hocker and R. Jackiw, Phys. Rev. Lett. **50**, 1719 (1983).

[3] F. Strocchi, *General Properties of Quantum Field Theory*, World Scientific, Singapore, 1993.

[4] A. S. Wightman, Phys. Rev. **101**, 860 (1956).

[5] R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and all that*, W. A. Benjamin, New York, 1964.

[6] M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Vol. II, Academic Press, New York, 1975.

[7] G. Baym, Phys. Rev. **117**, 886 (1960).
[8] F. J. Dyson, Phys. Rev. **85**, 631 (1952).
[9] L. Gårding and J. Lions, Nuovo Cimento Suppl. **14**, 9 (1959).
[10] G. Källén, Helv. Phys. Acta **25**, 417 (1952). H. Lehmann, Nuovo Cim. **11**, 342 (1954).
[11] P. G. Federbush and K. A. Johnson, Phys. Rev. **120**, 1926 (1960).
[12] R. Jost, *Properties of Wightman Functions* in *Lectures on Field Theory and the Many-Body Problem*, E. R. Caianiello (ed.), Academic Press, New York, 1961.
[13] B. Schroer, Phys. Lett. B **494**, 124 (2000).
[14] K. Pohlmeyer, Comm. Math. Phys. **12**, 201 (1969).
[15] H. Reeh und S. Schlieder, Nuovo Cimento **22**, 1051 (1961).
[16] We would like to thank H. Araki for drawing our attention to this point; H. Araki, private communications.
[17] J. Fröhlich, Nucl. Phys. **B 200** [**FS4**], 281 (1982).
[18] R. M. Wald, Comm. Math. Phys. **54**, 1 (1977).
[19] K. G. Wilson, Phys. Rev. **179**, 1499 (1969).
[20] S. Coleman and R. Jackiw, Ann. Phys. (N.Y.) **67**, 552 (1971).
[21] J. C. Collins, Phys. Rev. D **14**, 1965 (1976).
[22] These considerations will be established in detail in a forthcoming publication.
[23] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. B **147**, 385-447 (1979); Nucl. Phys. B **147**, 448-518 (1979).
[24] J. C. Collins, A. Duncan, and S. D. Joglekar, Phys. Rev. D **16**, 438 (1977).